Journal of Multivariate Analysis 101 (2010) 44–51

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

On hazard rate ordering of the sums of heterogeneous geometric random variables

Peng Zhao [a,*], Taizhong Hu [b]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
[b] Department of Statistics and Finance, University of Science and Technology of China, Hefei, Anhui 230026, China

ARTICLE INFO

Article history:
Received 23 December 2008
Available online 14 April 2009

AMS 2000 subject classifications:
primary 60E15
secondary 60K10

Keywords:
Stochastic order
Hazard rate order
Likelihood ratio order
Majorization
p-larger order
Negative binomial

ABSTRACT

In this paper, we treat convolutions of heterogeneous geometric random variables with respect to the p-larger order and the hazard rate order. It is shown that the p-larger order between two parameter vectors implies the hazard rate order between convolutions of two heterogeneous geometric sequences. Specially in the two-dimensional case, we present an equivalent characterization. The case when one convolution involves identically distributed variables is discussed, and we reveal the link between the hazard rate order of convolutions and the geometric mean of parameters. Finally, we drive the "best negative binomial bounds" for the hazard rate function of any convolution of geometric sequence under this setup.

1. Introduction

Because of its nice mathematical form and the memoryless property, the exponential distribution has been widely used in many areas, including life-testing, reliability, and operations research. One may refer to [1,2] for an encyclopedic treatment of developments on the exponential distribution. Convolutions of independent exponential random variables often occur naturally in many problems, and many researchers have investigated ordering properties based on exponential convolutions, including [3–8]. Also, we refer to [9–13] for ordering results of sums of independent random variables. The geometric random variable may be regarded as the discrete counterpart of the exponential random variable in the sense that both of them have constant hazard rate. In many practical situations and especially in reliability scenarios, convolutions of independent geometric random variables appear in a natural way; see [14,3] regarding some nice applications. This also motivates the present investigations. Since the distribution theory is quite complicated when the convolution involves non-identical random variables, it will be of great interest to derive bounds and approximations on some characteristics of interest in this setup.

In this paper, we investigate the ordering properties of the convolutions of heterogeneous geometric random variables. Let $X_{p_1}, \ldots, X_{p_n}$ and $X_{p_1^*}, \ldots, X_{p_n^*}$ be two sequences of independent geometric random variables with parameters $p_1, \ldots, p_n$ and $p_1^*, \ldots, p_n^*$, respectively. It is shown that for the two-dimensional case,

* Corresponding author.
E-mail addresses: zhaop07@gmail.com (P. Zhao), thu@ustc.edu.cn (T. Hu).

0047-259X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2009.04.002

$$
\begin{aligned}
& X_{p_1} + X_{p_2} \geq_{\text{hr}} X_{p_1^*} + X_{p_2^*} \\
\Longleftrightarrow\ & X_{p_1} + X_{p_2} \geq_{\text{st}} X_{p_1^*} + X_{p_2^*} \\
\Longleftrightarrow\ & (p_1, p_2) \overset{\text{p}}{\succeq} (p_1^*, p_2^*),
\end{aligned}
$$

and for the general case,

$$(p_1, \ldots, p_n) \overset{\text{p}}{\succeq} (p_1^*, \ldots, p_n^*) \Longrightarrow \sum_{i=1}^{n} X_{p_i} \geq_{\text{hr}} \sum_{i=1}^{n} X_{p_i^*},$$

where the formal definitions of related partial orderings will be given in Section 2. The main results for the two-dimensional case and the general case are given in Sections 3 and 4, respectively. Specially, we discuss the case when one convolution involves identically distributed random variables and show in this case that the hazard rate order is actually associated with the geometric mean of parameters. Finally, we derive the "best negative binomial bounds" for hazard rate function of any convolution of geometric distributions.

2. Definitions

In this section, we recall some notions of stochastic orders, and majorization and related orders which are closely related to the main results to be developed in what follows. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

Discrete stochastic orders

Definition 2.1. For two discrete random variables X and Y with common supports on integers $\mathbb{N}_0 \equiv \{0, 1, \ldots\}$, denote by $f_X(k)$ and $f_Y(k)$ their respective probability mass function (pmf), and $\overline{F}_X(k) = P(X \geq k)$ and $\overline{F}_Y(k) = P(Y \geq k)$ the corresponding survival functions. Then

(i) X is said to be smaller than Y in the likelihood ratio order, denoted by $X \leq_{\text{lr}} Y$, if $f_Y(k)/f_X(k)$ is increasing in $k \in \mathbb{N}_0$;
(ii) X is said to be smaller than Y in the hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if $\overline{F}_Y(k)/\overline{F}_X(k)$ is increasing in $k \in \mathbb{N}_0$;
(iii) X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{\text{st}} Y$, if $\overline{F}_Y(k) \geq \overline{F}_X(k)$ for all $k \in \mathbb{N}_0$.

It is well known that there exists the following implication relation:

$$X \leq_{\text{lr}} Y \Longrightarrow X \leq_{\text{hr}} Y \Longrightarrow X \leq_{\text{st}} Y.$$

For a comprehensive discussion on various stochastic orders, one may refer to [15,16].

Majorization and related orders

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.2. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$, written as $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

In addition, the vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly, written as $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Clearly,

$$\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}.$$

Those functions that preserve the majorization ordering are said to be Schur-convex. For extensive and comprehensive discussion on the theory and applications of the majorization order, one may refer to [17]. Bon and Păltănea [4] introduced a pre-order on $\mathbb{R}_+^n$, called p-larger order, which is defined as follows.

Definition 2.3. The vector $\mathbf{x}$ in $\mathbb{R}_+^n$ is said to be p-larger than another vector $\mathbf{y}$, written as $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$, if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Let $\log(\mathbf{x})$ be the vector of logarithms of the coordinates of $\mathbf{x}$. It is then easy to verify that

$$\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y} \Longleftrightarrow \log(\mathbf{x}) \overset{\text{w}}{\succeq} \log(\mathbf{y}).$$

Moreover,

$$\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$. The converse is, however, not true. For example, we have $(1, 5.5) \overset{\text{p}}{\succeq} (2, 3)$, but the weak majorization order clearly does not hold.

3. Equivalent characterization for the two-dimensional case

Let X_p be a geometric random variable with pmf

$$f_p(k) = P(X_p = k) = p(1-p)^k, \quad k \in \mathbb{N}_0,$$

and survival function

$$\overline{F}_p(k) = P(X_p \geq k) = (1-p)^k, \quad k \in \mathbb{N}_0.$$

Then the hazard rate function of X_p is given by

$$h_p(k) = \frac{f_p(k)}{\overline{F}_p(k)} = p, \quad k \in \mathbb{N}_0.$$

Let X_{p_1} and X_{p_2} be independent geometric random variables with respective parameters p_1 and p_2. If $p_1 \neq p_2$, then the pmf of $X_{p_1} + X_{p_2}$ can be written as

$$f_{(p_1,p_2)}(k) = P(X_{p_1} + X_{p_2} = k) = \frac{p_1 p_2[(1-p_1)^{k+1} - (1-p_2)^{k+1}]}{p_2 - p_1} \tag{1}$$

for $k \in \mathbb{N}_0$; and if $p_1 = p_2 = p$, $f_{(p_1,p_2)}(k)$ turns to be the pmf of a negative binomial distribution given by

$$f_{(p,p)}(k) = (k+1)p^2(1-p)^k, \quad k \in \mathbb{N}_0. \tag{2}$$

From Eqs. (1) and (2), the hazard rate function of $X_{p_1} + X_{p_2}$ is given by

$$h_{(p_1,p_2)}(k) = p_1 p_2 \frac{(1-p_1)^{k+1} - (1-p_2)^{k+1}}{p_2(1-p_1)^{k+1} - p_1(1-p_2)^{k+1}}, \quad p_1 \neq p_2, \tag{3}$$

and

$$h_{(p,p)}(k) = \frac{(k+1)p^2}{kp+1}, \quad p_1 = p_2 = p, \tag{4}$$

where $k \in \mathbb{N}_0$. Under the geometric framework, this section establishes an equivalent characterization between the p-larger order of parameter vectors and the hazard rate order (the usual stochastic order) of convolutions for the two-dimensional case, which are not only helpful for deriving our main results in the next section, but are also of independent interest. The following lemma, due to [17], will be used to establish Theorem 3.2 below.

Lemma 3.1 ([17, p. 57]). *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Then ϕ is Schur-convex on I^n if and only if ϕ is symmetric on I^n and for all $i \neq j$,*

$$(z_i - z_j)\left[\frac{\partial}{\partial z_i}\phi(\boldsymbol{z}) - \frac{\partial}{\partial z_j}\phi(\boldsymbol{z})\right] \geq 0 \quad \text{for all } \boldsymbol{z} \in I^n,$$

where $\frac{\partial}{\partial z_i}\phi(\boldsymbol{z})$ denotes the partial derivative with respect to its ith argument.

Assume $X_{\lambda_1}, X_{\lambda_2}$ and $X_{\lambda_1^*}, X_{\lambda_2^*}$ to be two pairs of independent exponential random variables with hazard rates λ_1, λ_2 and λ_1^*, λ_2^*, respectively. Bon and Păltănea [4] then proved the following equivalence:

$$X_{\lambda_1} + X_{\lambda_2} \geq_{\text{hr}}[\geq_{\text{st}}] X_{\lambda_1^*} + X_{\lambda_2^*} \Longleftrightarrow (\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*). \tag{5}$$

The result established below presents an equivalent characterization similar to (5) under the geometric setup.

Theorem 3.2. *Let (X_{p_1}, X_{p_2}) be a vector of independent geometric random variables with respective parameters p_1, p_2, and $(X_{p_1^*}, X_{p_2^*})$ be another vector of independent geometric random variables with respective parameters p_1^*, p_2^*. Then the following three statements are equivalent:*

(i) $(p_1, p_2) \overset{\mathrm{p}}{\succeq} (p_1^*, p_2^*)$;
(ii) $X_{p_1} + X_{p_2} \geq_{\mathrm{hr}} X_{p_1^*} + X_{p_2^*}$;
(iii) $X_{p_1} + X_{p_2} \geq_{\mathrm{st}} X_{p_1^*} + X_{p_2^*}$.

Proof. It is well known that the hazard rate order implies the usual stochastic order and, hence, we only need to show (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). Without loss of generality, assume that $p_1 \leq p_2$ and $p_1^* \leq p_2^*$.

(i) $\Rightarrow$ (ii) Note that, if $p_1p_2 < p_1^*p_2^*$, then there exists some p_2' such that $p_2 < p_2'$ and $p_1p_2' = p_1^*p_2^*$. From Lemma 1.B.3 of [15], it follows immediately that $X_{p_1} + X_{p_2} \geq_{\mathrm{hr}} X_{p_1} + X_{p_2'}$. As a result, we find that it is enough to establish the necessity under the following condition:

$$p_1 \leq p_1^* \quad \text{and} \quad p_1p_2 = p_1^*p_2^*.$$

Moreover, if $p_1 = p_1^*$, then $p_2 = p_2^*$ and this becomes a trivial case. Now we proceed to establish the required result under the condition $p_1 < p_1^* \leq p_2^* < p_2$ and $p_1p_2 = p_1^*p_2^*$. In fact, it can be checked that the hazard rate functions in (3) and (4) have a unitary formula as follows:

$$h_{(p_1,p_2)}(k) = p_1p_2 \frac{\sum_{i=0}^{k} (1-p_1)^i (1-p_2)^{k-i}}{(1-p_1)^{k+1} + p_1 \sum_{i=0}^{k} (1-p_1)^i (1-p_2)^{k-i}} \tag{6}$$

for all $p_1, p_2 \in (0, 1)$ and $k \in \mathbb{N}_0$. We need to show that

$$h_{(p_1,p_2)}(k) \leq h_{(p_1^*,p_2^*)}(k), \quad k \in \mathbb{N}_0. \tag{7}$$

Clearly,

$$h_{(p_1,p_2)}(0) = p_1p_2 = p_1^*p_2^* = h_{(p_1^*,p_2^*)}(0).$$

Suppose $k \geq 1$. Denote $x = \log p_1, y = \log p_2, x^* = \log p_1^*$ and $y^* = \log p_2^*$. Since $p_1, p_2, p_1^*, p^* \in (0, 1)$, we have $x, y, x^*, y^* < 0$. Also, the following relation holds:

$$(x, y) \overset{\mathrm{m}}{\succeq} (x^*, y^*).$$

We then observe that it suffices to show that the symmetrical conditional differentiable function $H_k : (-\infty, 0)^2 \to (0, \infty)$, defined by

$$H_k(x, y) = \mathrm{e}^x + \frac{(1-\mathrm{e}^x)^{k+1}}{\sum_{i=0}^{k} (1-\mathrm{e}^y)^i (1-\mathrm{e}^x)^{k-i}} = \mathrm{e}^y + \frac{(1-\mathrm{e}^y)^{k+1}}{\sum_{i=0}^{k} (1-\mathrm{e}^x)^i (1-\mathrm{e}^y)^{k-i}}, \quad \forall x, y < 0,$$

is Schur-convex. Note that

$$\frac{\partial}{\partial x} H_k(x, y) - \frac{\partial}{\partial y} H_k(x, y) = \frac{\sum_{i=1}^{k} i(1-\mathrm{e}^x)^{i-1}(1-\mathrm{e}^y)^{i-1}[\mathrm{e}^x(1-\mathrm{e}^y)^{2(k+1-i)} - \mathrm{e}^y(1-\mathrm{e}^x)^{2(k+1-i)}]}{\left[\sum_{i=0}^{k} (1-\mathrm{e}^x)^i (1-\mathrm{e}^y)^{k-i}\right]^2}.$$

It is easy to verify that

$$(x - y)\left[\mathrm{e}^x(1-\mathrm{e}^y)^{2(k+1-i)} - \mathrm{e}^y(1-\mathrm{e}^x)^{2(k+1-i)}\right] \geq 0$$

for all $x, y < 0$ and $i = 1, 2, \ldots, k$, which implies that

$$(x - y)\left(\frac{\partial}{\partial x} H_k(x, y) - \frac{\partial}{\partial y} H_k(x, y)\right) \geq 0$$

for all $x, y < 0$. Upon using Lemma 3.1, it follows that H_k is Schur-convex on $(-\infty, 0)^2$ for $k = 1, 2, \ldots$. We therefore prove the inequality in (7).

(iii) $\Rightarrow$ (i) Suppose $X_{p_1} + X_{p_2} \geq_{\mathrm{st}} X_{p_1^*} + X_{p_2^*}$, i.e., $\overline{F}_{(p_1,p_2)}(k) \geq \overline{F}_{(p_1^*,p_2^*)}(k)$ for all $k \in \mathbb{N}_0$, which implies

$$\overline{F}_{(p_1,p_2)}(1) - \overline{F}_{(p_1^*,p_2^*)}(1) = (1 - p_1p_2) - (1 - p_1^*p_2^*) \geq 0,$$

and hence it holds that $p_1p_2 \le p_1^*p_2^*$. Next we will show $p_1 \le p_1^*$. Assume that $p_1 \ne p_2$ and $p_1^* \ne p_2^*$. It can then be verified that

$$\lim_{k\to\infty} \frac{\overline{F}_{(p_1,p_2)}(k)}{\overline{F}_{(p_1^*,p_2^*)}(k)} = \lim_{k\to\infty} \frac{p_1p_2(p_2^*-p_1^*)}{(p_2-p_1)p_1^*p_2^*} \cdot \frac{\frac{1}{p_1}-\frac{1}{p_2}\left[\frac{1-p_2}{1-p_1}\right]^{k+1}}{\frac{1}{p_1^*}-\frac{1}{p_2^*}\left[\frac{1-p_2^*}{1-p_1^*}\right]^{k+1}} \cdot \left[\frac{1-p_1}{1-p_1^*}\right]^{k+1}.$$

This means that $\lim_{k\to\infty} \overline{F}_{(p_1,p_2)}(k)/\overline{F}_{(p_1^*,p_2^*)}(k) = 0$ if $p_1 > p_1^*$, in contradiction with the fact that $\overline{F}_{(p_1,p_2)}(k)/\overline{F}_{(p_1^*,p_2^*)}(k) \ge 1$ for all $k \in \mathbb{N}_0$. We therefore obtain that $p_1 \le p_1^*$. For the case either $p_1 = p_2$ or $p_1^* = p_2^*$, the same conclusion can be deduced by using a limiting argument. This completes the proof of the theorem. ∎

Remark 3.3. Boland et al. [3] gave two sufficient conditions on the parameters of underlying distributions under which two convolutions of independent geometric random variables can be ordered with respect to the likelihood ratio order. It should be pointed out that they used a different parametrization of the geometric distribution from ours. More precisely, under the setup of Theorem 3.2, Boland et al. [3] proved that if

$$(p_1, p_2) \overset{\mathrm{m}}{\succeq} (p_1^*, p_2^*) \tag{8}$$

or if

$$(\log(1-p_1), \log(1-p_2)) \overset{\mathrm{m}}{\succeq} \left(\log(1-p_1^*), \log(1-p_2^*)\right), \tag{9}$$

then

$$X_{p_1} + X_{p_2} \ge_{\mathrm{lr}} X_{p_1^*} + X_{p_2^*}.$$

Obviously, condition (8) implies that $(p_1, p_2) \overset{\mathrm{w}}{\succeq} (p_1^*, p_2^*)$. By Theorems 5.A.1 and 5.A.2 of [17], it is seen that condition (9) also implies $(p_1, p_2) \overset{\mathrm{w}}{\succeq} (p_1^*, p_2^*)$, which in turn implies $(p_1, p_2) \overset{\mathrm{p}}{\succeq} (p_1^*, p_2^*)$. The latter is exactly the equivalent characterization for the hazard rate ordering of convolutions. However, $(p_1, p_2) \overset{\mathrm{p}}{\succeq} (p_1^*, p_2^*)$ does not imply condition (8) or (9). So Theorem 3.2 cannot be derived from the main results in [3].

4. HR ordering between convolutions of geometric distributions

The following result presents a natural extension of Theorem 3.2 from the two-dimensional case to the general case and it should be mentioned here that we prove it by using the same technique as in [4].

Theorem 4.1. *Let $X_{p_1}, \ldots, X_{p_n}$ and $X_{p_1^*}, \ldots, X_{p_n^*}$ be two sequences of independent geometric random variables with parameters $p_1, \ldots, p_n$ and $p_1^*, \ldots, p_n^*$, respectively.*

(1) *If $(p_1, \ldots, p_n) \overset{\mathrm{p}}{\succeq} (p_1^*, \ldots, p_n^*)$, then*

$$\sum_{i=1}^n X_{p_i} \ge_{\mathrm{hr}} \sum_{i=1}^n X_{p_i^*}.$$

(2) *Conversely, if $\sum_{i=1}^n X_{p_i} \ge_{\mathrm{st}} \sum_{i=1}^n X_{p_i^*}$, then*

$$\min\{p_1, \ldots, p_n\} \le \min\{p_1^*, \ldots, p_n^*\} \quad \textit{and} \quad \prod_{i=1}^n p_i \le \prod_{i=1}^n p_i^*.$$

Proof. Without loss of generality, let us assume that $p_1 \le \cdots \le p_n$ and $p_1^* \le \cdots \le p_n^*$.

(1) Suppose that $(p_1, \ldots, p_n) \overset{\mathrm{p}}{\succeq} (p_1^*, \ldots, p_n^*)$, that is, $\prod_{i=1}^j p_i \le \prod_{i=1}^j p_i^*$ for $j = 1, \ldots, n$. The proof follows by induction on n. The result is trivially true for $n = 1$. For $n = 2$, it follows from Theorem 3.2. We assume the result to hold for $n-1$ $(n \ge 3)$ and proceed to establish it for n. To conclude, let us distinguish two cases.

Case (i): $p_1^* < p_n$

In this case, we have $p_1 \le p_1^* < p_n$ and, hence, there must exist exactly some integer k $(1 \le k \le n-1)$ such that $p_k \le p_1^* < p_{k+1}$. Since

$$(p_k, p_{k+1}) \overset{\mathrm{p}}{\succeq} \left(p_1^*, \frac{p_kp_{k+1}}{p_1^*}\right),$$

it follows from Theorem 3.2 that

$$X_{p_k} + X_{p_{k+1}} \geq_{\mathrm{hr}} X_{p_1^*} + X_{\frac{p_k p_{k+1}}{p_1^*}}, \tag{10}$$

where $X_{\frac{p_k p_{k+1}}{p_1^*}}$ denotes a geometric random variable with parameter $\frac{p_k p_{k+1}}{p_1^*}$, independent of all X_{p_i} and $X_{p_i^*}$. Also, it can be readily observed that

$$\left(p_1, \ldots, p_{k-1}, \frac{p_k p_{k+1}}{p_1^*}, p_{k+2}, \ldots, p_n\right) \overset{\mathrm{p}}{\succeq} (p_2^*, \ldots, p_k^*, p_{k+1}^*, \ldots, p_n^*).$$

Applying the induction assumption yields the following ordering inequality:

$$X_{p_1} + \cdots + X_{p_{k-1}} + X_{\frac{p_k p_{k+1}}{p_1^*}} + X_{p_{k+2}} + \cdots + X_{p_n} \geq_{\mathrm{hr}} X_{p_2^*} + \cdots + X_{p_k^*} + X_{p_{k+1}^*} + \cdots + X_{p_n^*}. \tag{11}$$

Notice that a convolution of independent geometric random variables possesses IFR property. Upon using the discrete version of Theorem 1.B.4 of [15] and the inequalities (10) and (11), we conclude

$$\sum_{i=1}^{n} X_{p_i} \geq_{\mathrm{hr}} \sum_{i=1}^{k-1} X_{p_i} + X_{p_1^*} + X_{\frac{p_k p_{k+1}}{p_1^*}} + \sum_{i=k+2}^{n} X_{p_i} \geq_{\mathrm{hr}} \sum_{i=1}^{n} X_{p_i^*}.$$

Case (ii): $p_1^* \geq p_n$

Since $p_i \leq p_i^*$ for all $1 \leq i \leq n$ in this case, the result is then a direct application of Theorem 1.B.4 of [15].

(2) First assume that $p_i \neq p_j$ and $p_i^* \neq p_j^*$ for all $i \neq j$. Then the pmf of the convolution $T_n = \sum_{i=1}^{n} X_{p_i}$ is given by (see [14]),

$$f_{T_n}(k) = \sum_{i=1}^{n} p_i (1-p_i)^{n+k-1} \prod_{j=1, j\neq i}^{n} \left(\frac{p_j}{p_j - p_i}\right), \quad k \in \mathbb{N}_0, \tag{12}$$

and the corresponding survival function can be written as

$$\overline{F}_{T_n}(k) = \sum_{i=1}^{n} (1-p_i)^{n+k-1} \prod_{j=1, j\neq i}^{n} \left(\frac{p_j}{p_j - p_i}\right), \quad k \in \mathbb{N}_0.$$

Denote $T_n^* = \sum_{i=1}^{n} X_{p_i^*}$. Then we have

$$\begin{aligned} \frac{\overline{F}_{T_n}(k)}{\overline{F}_{T_n^*}(k)} &= \frac{\sum\limits_{i=1}^{n} (1-p_i)^{n+k-1} \prod\limits_{j=1, j\neq i}^{n} \left(\frac{p_j}{p_j - p_i}\right)}{\sum\limits_{i=1}^{n} (1-p_i^*)^{n+k-1} \prod\limits_{j=1, j\neq i}^{n} \left(\frac{p_j^*}{p_j^* - p_i^*}\right)} \\ &= \left[\frac{1-p_1}{1-p_1^*}\right]^{n+k+1} \cdot \frac{\prod\limits_{j=2}^{n} \left(\frac{p_j}{p_j - p_1}\right) + \sum\limits_{i=2}^{n} \left[\frac{1-p_i}{1-p_1}\right]^{n+k-1} \prod\limits_{j=1, j\neq i}^{n} \left(\frac{p_j}{p_j - p_i}\right)}{\prod\limits_{j=2}^{n} \left(\frac{p_j^*}{p_j^* - p_1^*}\right) + \sum\limits_{i=2}^{n} \left[\frac{1-p_i^*}{1-p_1^*}\right]^{n+k-1} \prod\limits_{j=1, j\neq i}^{n} \left(\frac{p_j^*}{p_j^* - p_i^*}\right)}. \end{aligned}$$

Note that, if $p_1 > p_1^*$ then $\lim_{k\to\infty} \overline{F}_{T_n}(k)/\overline{F}_{T_n^*}(k) = 0$, contradicting the fact that $\overline{F}_{T_n}(k) \geq \overline{F}_{T_n^*}(k)$ for all $k \in \mathbb{N}_0$. We thus have $p_1 \leq p_1^*$. For the case that either p_i or p_i^* are not pairwise unequal, the desired result can also be obtained by using a limiting argument.

On the other hand, observe that

$$\overline{F}_{T_n}(1) = 1 - P(T_n = 0) = 1 - \prod_{i=1}^{n} p_i.$$

Similarly, $\overline{F}_{T_n^*}(1) = 1 - \prod_{i=1}^{n} p_i^*$. The usual stochastic order assumption implies that

$$\overline{F}_{T_n}(1) = 1 - \prod_{i=1}^{n} p_i \geq 1 - \prod_{i=1}^{n} p_i^* = \overline{F}_{T_n^*}(1),$$

which is equivalent to $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$. This thus completes the proof. ■

Remark 4.2. Under the setup of Theorem 4.1, Boland et al. [3] actually proved that if

$$(p_1, \dots, p_n) \overset{\mathrm{m}}{\succeq} (p_1^*, \dots, p_n^*) \tag{13}$$

or if

$$(\log(1-p_1), \dots, \log(1-p_n)) \overset{\mathrm{m}}{\succeq} \left(\log(1-p_1^*), \dots, \log(1-p_n^*)\right), \tag{14}$$

then

$$\sum_{i=1}^{n} X_{p_i} \geq_{\mathrm{lr}} \sum_{i=1}^{n} X_{p_i^*}.$$

Again, by Theorems 5.A.1 and 5.A.2 of [17], it is seen that either one of conditions (13) and (14) is stronger than the one in the first part of Theorem 4.1.

In what follows, we turn to discussing the special case wherein one convolution involves i.i.d. geometric random variables. As a direct application of Theorem 4.1, we can readily obtain the following corollary, which reveals the connection between the hazard rate ordering of the convolutions and the geometric mean of their parameters. Moreover, the equivalences in Corollary 4.3 in fact correspond to those of Corollary and Theorem 2 in [4] for the exponential case.

Corollary 4.3. *Let $X_{p_1}, \dots, X_{p_n}$ be independent geometric random variables with respective parameters $p_1, \dots, p_n$, and $Y_1, \dots, Y_n$ be an i.i.d. geometric sequence with a common parameter p. Denote*

$$T_n(p_1, \dots, p_n) = \sum_{i=1}^{n} X_{p_i} \quad \text{and} \quad T_n(p, \dots, p) = \sum_{i=1}^{n} Y_i.$$

Then

$$T_n(p_1, \dots, p_n) \geq_{\mathrm{hr}} [\geq_{\mathrm{st}}]\, T_n(p, \dots, p) \Longleftrightarrow p \geq \sqrt[n]{p_1 \cdots p_n}$$

and

$$T_n(p_1, \dots, p_n) \leq_{\mathrm{hr}} [\leq_{\mathrm{st}}]\, T_n(p, \dots, p) \Longleftrightarrow p \leq \min\{p_1, \dots, p_n\}.$$

Actually, $T_n(p, \dots, p)$ is a negative binomial random variable with parameters (n, p) and its pmf is given by

$$f_{(n,p)}(k) = \binom{k+n-1}{n-1} p^n (1-p)^k, \quad k \in \mathbb{N}_0.$$

Let $h_{(n,p)}(k)$ be the hazard rate function of the negative binomial random variable $T_n(p, \dots, p)$, and let $h_{(p_1,\dots,p_n)}(k)$ be the hazard rate function of the convolution $T_n(p_1, \dots, p_n)$. We then obtain from Corollary 4.3 that the best negative binomial bounds for $h_{(p_1,\dots,p_n)}(k)$ as follows:

$$h_{(n, \min_{1\leq i\leq n} p_i)}(k) \leq h_{(p_1,\dots,p_n)}(k) \leq h_{\left(n, \sqrt[n]{p_1 \cdots p_n}\right)}(k) \tag{15}$$

for all $k \in \mathbb{N}_0$.

We next present a numerical example for illustrating the best negative binomial bounds established above. For the case $n = 3$, we have

$$h_{(p_1,p_2,p_3)}(k) = \frac{\sum_{(i,j,k)\in\mathscr{P}_3} p_i(1-p_i)^{k+2} \cdot \frac{p_j p_k}{(p_j-p_i)(p_k-p_i)}}{\sum_{(i,j,k)\in\mathscr{P}_3} (1-p_i)^{k+2} \cdot \frac{p_j p_k}{(p_j-p_i)(p_k-p_i)}}$$

and

$$h_{(3,p)}(k) = \frac{(k+2)(k+1)p^3}{(k+2)(k+1)p^2 + 2(k+2)p(1-p) + 2(1-p)^2}$$

for $k \in \mathbb{N}_0$, where $\mathscr{P}_3$ is the permutation group of the vector $(1, 2, 3)$. Upon choosing $(p_1, p_2, p_3) = (1/8, 1/4, 1/2)$, the arithmetic mean and geometric mean are given by $(p_1 + p_2 + p_3)/3 = 7/24$, $\sqrt[3]{p_1 p_2 p_3} = 1/4$, respectively, and $\min\{p_1, p_2, p_3\} = 1/8$. As shown in Fig. 1, four plots of the functions

$$h_{(3,1/8)}(k), \qquad h_{(1/8,1/4,1/2)}(k), \qquad h_{(3,1/4)}(k), \quad \text{and} \quad h_{(3,7/24)}(k)$$

are successively ordered from the bottom to the top, which is exactly in accordance with (15). It can be readily seen that, for the hazard rate function $h_{(1/8,1/4,1/2)}(k)$, the negative binomial upper bound $h_{(3,1/4)}(k)$ is the best approximation near the origin, while the negative binomial lower bound $h_{(3,1/8)}(k)$ is the best approximation in the tail.

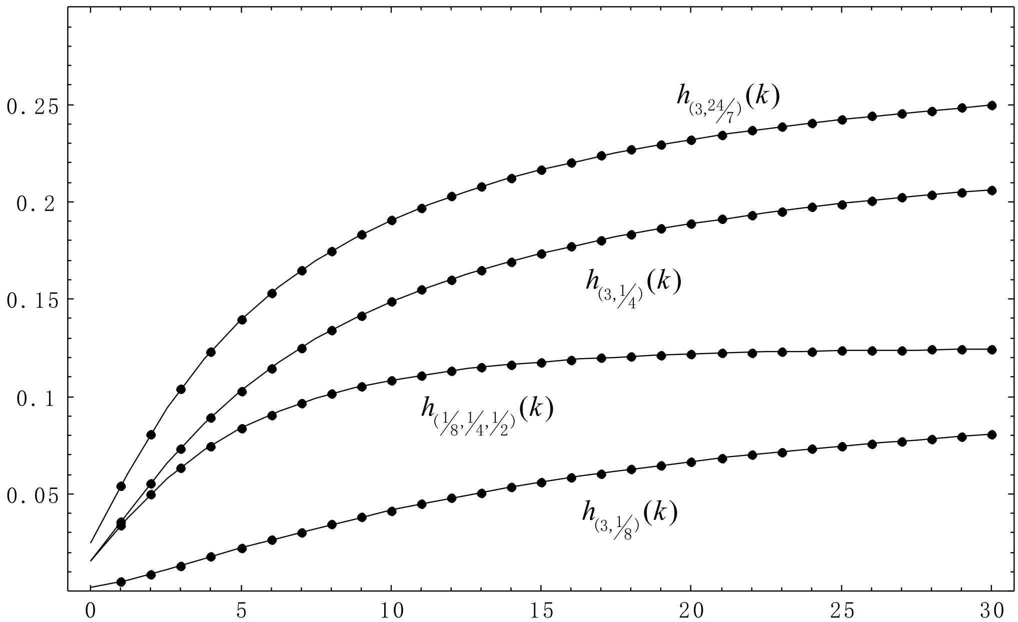

Fig. 1. Plot of the hazard rate functions of the convolution with parameters (1/8, 1/4, 1/2) and of negative binomial distributions with parameters as arithmetic mean, geometric mean and 1/8, respectively.

Acknowledgments

The authors would like to thank the referees for their insightful suggestions which led to a considerable improvement in the presentation and quality of this manuscript. Especially, the novel and brief proof of the first part of Theorem 3.2 was supplied by one of the referees which substantially simplified the proof of the original version.

Second author was supported by the National Natural Science Foundations of China (No. 70821001,10771204).

References

[1] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing, To Begin With, Silver Spring, Maryland, 1981.
[2] N. Balakrishnan, A.P. Basu (Eds.), The Exponential Distribution: Theory, Methods and Applications, Gordon and Breach, Newark, New Jersey, 1995.
[3] P.J. Boland, E. El-Neweihi, F. Proschan, Schur properties of convolutions of exponential and geometric random variables, Journal of Multivariate Analysis 48 (1994) 157–167.
[4] J.L. Bon, E. Păltănea, Ordering properties of convolutions of exponential random variables, Lifetime Data Analysis 5 (1999) 185–192.
[5] S.C. Kochar, C. Ma, Dispersive ordering of convolutions of exponential random variables, Statistics & Probability Letters 43 (1999) 321–324.
[6] R.M. Korwar, On stochastic orders for sums of independent random variables, Journal of Multivariate Analysis 80 (2002) 344–357.
[7] B.-E. Khaledi, S.C. Kochar, Ordering convolutions of gamma random variables, Sankhyā 66 (2004) 466–473.
[8] P. Zhao, N. Balakrishnan, Mean residual life order of convolutions of heterogeneous exponential random variables, Journal of Multivariate Analysis (2009) doi:10.1016/j.jmva.2009.02.009.
[9] P.J. Boland, F. Proschan, Y.L. Tong, A stochastic ordering of partial sums of independent random variables and of some random processes, Journal of Applied Probability 29 (1992) 645–654.
[10] P.J. Boland, H. Singh, Stochastic comparisons of Bernoulli sums and Binomial random variables, in: Advances in Distribution Theory, Order Statistics, and Inference, Stat. Ind. Technol., Birkhauser Boston, Boston, MA, 2006, pp. 3–11.
[11] P.J. Boland, The probability distribution for the number of successes in independent trials, Communications in Statistics-Theory and Methods 36 (2007) 1327–1331.
[12] M. Denuit, C. Genest, E. Marceau, Criteria for the stochastic ordering of random sums, with actuarial applications, Scandinavian Actuarial Journal 1 (2002) 3–16.
[13] C. Ma, A note on stochastic ordering of order statistics, Journal of Applied Probability 34 (1997) 785–789.
[14] A. Sen, N. Balakrishnan, Convolution of geometrics and a reliability problem, Statistics & Probability Letters 43 (1999) 421–426.
[15] M. Shaked, J.G. Shanthikumar, Stochastic Orders, Springer-Verlag, New York, 2007.
[16] A. Müller, D. Stoyan, Comparison Methods for Stochastic Models and Risks, John Wiley & Sons, New York, 2002.
[17] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.